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Set-open topologies on function spaces

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Abstract

Let X and Y be topological spaces, $F(X,Y)$ the set of all functions from X into Y and $C(X,Y)$ the set of all continuous functions in $F(X,Y)$. We study various set-open topologies t_λ ($\lambda \subseteq \mathcal{P}(X)$) on $F(X,Y)$ and consider their existence, comparison and coincidence in the setting of Y a general topological space as well as for $Y = \mathbb{R}$. Further, we consider the parallel notion of quasi-uniform convergence topologies $\mathcal{U}_\lambda$ ($\lambda \subseteq \mathcal{P}(X)$) on $F(X,Y)$ to discuss $\mathcal{U}_\lambda$-closedness and right $\mathcal{U}_\lambda$-K-completeness properties of a certain subspace of $F(X,Y)$ in the case of Y a locally symmetric quasi-uniform space. We include some counter-examples to justify our comments.

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1. Introduction

One of the original set-open topologies on $C(X,Y)$ is the compact-open topology t_k, which was introduced by Fox [7] and further developed by Arens [2], Gale [8], Myers [24], Arens-Dugundji [3] and Jackson [12]. Later, many

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other set-open topologies t_λ ($\lambda \subseteq P(X)$) were investigated that lie between t_k and t_w (the largest set-open topology) (see, e.g., [9, 10, 15, 17, 5, 27, 29]).

Apart from the set-open topologies, there is also a parallel notion of "uniform convergence topologies" $\mathcal{U}_\lambda$ ($\lambda \subseteq \mathcal{P}(X)$) on $F(X,Y)$ which were discussed in datail by Kelley [13] in the case of $Y = (Y,\mathcal{U})$ a uniform space and by Naimpally [25] in the case of $Y = (Y,\mathcal{U})$ a quasi-uniform space. These have been further investigated by several authors, including Papadopoulos [34], Kunzi and Romaguera [20] and more recently in [1]. These topologies $\mathcal{U}_\lambda$ ($\lambda \subseteq \mathcal{P}(X)$) are, in general, different from their corresponding set-open topologies t_λ ($\lambda \subseteq \mathcal{P}(X)$) even in the case of Y a metric space, but the two notions coincide in some particular cases (see [4, 13, 17, 23, 28, 29, 33]).

In this paper, we study various set-open topologies on $F(X,Y)$ in the setting of X and Y arbitrary topological spaces. In section 2, we study their comparability and also coincidence of such topologies; we also discuss their existence and their relationship with some uniform convergence topologies. In section 3, we establish some results on closedness and completeness of the space $C^\alpha(X,Y)$ of all α-continuous functions, from X into Y [16, 21]. Here, we shall need to assume that Y is a regular topological space, which is equivalent to Y being a locally symmetric quasi-uniform space [6, 25].

2. Set-open Topologies on $F(X,Y)$

Recall that the space X is said to be **pseudocompact** if every $f \in C(X)$ is bounded on X. A subset A of X is called **C-compact** (resp. **bounded)** if the set $f(A)$ is compact (resp. bounded) in $\mathbb{R}$ for every $f \in C(X)$. If $A = X$, the property of the set A to be C-compact (bounded) coincides with the pseudocompactness of X.

Notations. For any topological space X, let $\mathcal{P}(X)$ denote the power set of X, and let

$$\begin{aligned}
\mathcal{F}(X) &= \{A \subseteq X : A \text{ is finite}\},\\
\mathcal{K}(X) &= \{A \subseteq X : A \text{ is compact}\},\\
\mathcal{PS}(X) &= \{A \subseteq X : A \text{ is pseudocompact}\},\\
\mathcal{RC}(X) &= \{A \subseteq X : A \text{ is } C\text{-compact}\}.
\end{aligned}$$

Clearly, $\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{PS}(X) \subseteq \mathcal{RC}(X)$.

Recall that a collection $\lambda \subseteq \mathcal{P}(X)$ is called a **network** on X if, for each $x \in X$ and each open neighborhood U of x, there exists an $A \in \lambda$ such that $x \in A \subseteq U$. A network λ on X is called a **closed network** on X if each $A \in \lambda$ is closed. Since, for each $x \in X$ and each open neighborhood U of x, $x \in \{x\} \subseteq U$, it is clear that each of the collections $\mathcal{F}(X), K(X), \mathcal{PS}(X), \mathcal{RC}(X), \mathcal{P}(X)$ is a network on X. If X is a Hausdorff space, then $\mathcal{F}(X), \mathcal{K}(X)$ are closed networks.

Definition 2.1 (cf. [22, 5]). Let X and Y be topological spaces, and let $\lambda \subseteq \mathcal{P}(X)$ be a network which covers X. For any $A \in \lambda$ and open $G \subseteq Y$, let

$$N(A,G) = \{f \in F(X,Y) : f(A) \subseteq G\}.$$

Then the collection $\{N(A,G) : A \in \lambda, \text{ open } G \subseteq Y\}$ forms a subbase for a "set-open" topology on $F(X,Y)$, called the **λ-open topology** (cf. [3], p. 13) and denoted by t_λ; see also [7, 2, 13, 23]).

Using this terminology, we can consider the following topologies on $F(X,Y)$:

(1) $\mathcal{F}(X)$-open topology, the usual **point-open topology** t_p.
(2) $\mathcal{K}(X)$-open topology, the usual **compact-open topology** t_k.
(3) $\mathcal{PS}(X)$-open topology, the **pseudocompact-open topology** t_{ps} [18].
(4) $\mathcal{RC}(X)$-open topology, the **C-compact-open topology** t_{rc} [29, 30, 31, 32, 33].

If X and Y are any two spaces with the same underlying set, then we use $X = Y, X \le Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the spaces of real numbers and natural numbers, respectively. For convenience, we shall some times denote $(F(X,Y), t_\lambda)$ by $F_\lambda(X,Y)$.

If (X,τ) is a topological space and $A \subseteq X$, the closure of A is denoted by $\overline{A}$ or τ-$cl(A)$; the interior of A is denoted by $int(A)$ or τ-$int(A)$.

Lemma 2.2. *Let X and Y be non-empty sets. Suppose that $A, B \subseteq X$, and let $G, H \subseteq Y$ be non-empty sets such that $N(B,H) \subseteq N(A,G)$. Then:*

(i) If $A \neq \varnothing$, then $H \subseteq G$.
(ii) If $G \neq Y$, then $A \subseteq B$.

Proof. (i) Suppose $A \neq \varnothing$, but $H \nsubseteq G$, and let $c \in H \backslash G$. Consider the constant function $f_c : X \to Y$ defined by $f_c(x) = c$ $(x \in X)$. If $B = \varnothing$, then $f_c(B) = \varnothing \subseteq H$; if $B \neq \varnothing$, then $f_c(B) = \{c\} \subseteq H$. Hence $f_c \in N(B,H)$. Since $A \neq \varnothing$, $f_c(A) = \{c\} \nsubseteq G$ and so $f_c \notin N(A,G)$. This contradicts $N(B,H) \subseteq N(A,G)$. Therefore $H \subseteq G$.

(ii) Suppose $A \nsubseteq B$, and let $x_0 \in A \backslash B$. Since $G \neq Y$, choose $a \in Y \backslash G$. Let $p \in H$. Define $g = g_{B,p} : X \to Y$ by

$$\begin{aligned} g(x) &= p \text{ if } x \in B, \\ g(x) &= a \text{ if } x \in X \backslash B. \end{aligned}$$

Then $g \in F(X,Y)$. If $B = \varnothing$, then $g(B) = \varnothing \subseteq H$; if $B \neq \varnothing$, then $g(B) = \{p\} \subseteq H$. Hence $g \in N(B,H)$. Since $x_0 \in A \backslash B \subseteq X \backslash B$, $g(x_0) = a \notin G$, $g \notin N(A,G)$. This contradicts $N(B,H) \subseteq N(A,G)$. Therefore $A \subseteq B$. $\square$

Theorem 2.3. *Let X be a Hausdorff topological space and Y any topological space. Then:*

(a) $F_p(X,Y) \le F_k(X,Y) \le F_{ps}(X,Y) \le F_{rc}(X,Y)$.
(b) *$F_k(X,Y) = F_{ps}(X,Y)$ iff every closed pseudocompact subset of X is compact.*
(c) *$F_k(X,Y) = F_{rc}(X,Y)$ iff every closed C-compact subset of X is compact.*

(d) *$F_p(X,Y) = F_{rc}(X,Y)$ iff every C-compact subset of X is finite. In particular, if X is discrete, then $F_p(X,Y) = F_k(X,Y) = F_{ps}(X,Y) = F_{rc}(X,Y)$.*

Proof. (a) Since $\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{PS}(X) \subseteq \mathcal{RC}(X)$, it follows that $F_p(X,Y) \leq F_k(X,Y) \leq F_{ps}(X,Y) \leq F_{rc}(X,Y)$.

(b) Suppose $F_{ps}(X,Y) \leq F_k(X,Y)$, and let $H \subseteq Y$ be open and F be any closed pseudocompact subset of X, with $H \neq Y$.

Let $f_c(x) = c$ $(x \in X)$ where $c \in H$. Then there exist a compact subsets $K_1, ..., K_n$ of X and open subsets $G_1, ..., G_n$ of Y such that $f_c \in \bigcap\limits_{i=1}^{n} N(K_i, G_i) \subseteq N(F,H)$. Consider $K = \bigcup\limits_{i=1}^{n} K_i$ and $G = \bigcap\limits_{i=1}^{n} G_i$, then

$$N(K,G) \subseteq N(F,H).$$

By Lemma 2.2(ii), $F \subseteq K$. Thus, F is compact.

Conversely, suppose that every closed pseudocompact subset of X is compact. Note that for a subset A of X, $N(\overline{A},G) \subseteq N(A,G)$. It follows that $F_{ps}(X,Y) \leq F_k(X,Y)$.

(c) This can be proved in a manner similar to (b).

(d) Suppose $F_{rc}(X,Y) \leq F_p(X,Y)$, and let $K \subseteq X$ be a C-compact subset of X and $H \subseteq Y$ be open, with $H \neq Y$.

Let $f_c(x) = c$ $(x \in X)$ where $c \in H$. Then there exist a finite subsets $F_1, ..., F_n$ of X and open subsets $G_1, ..., G_n$ of Y such that $f_c \in \bigcap\limits_{i=1}^{n} N(F_i, G_i) \subseteq N(K,H)$. Consider $F = \bigcup\limits_{i=1}^{n} K_i$ and $G = \bigcap\limits_{i=1}^{n} G_i$, then

$$N(F,G) \subseteq N(K,H).$$

By Lemma 2.2(ii), $K \subseteq F$; hence K is finite.

Conversely, suppose that every C-compact subset of X is finite. To show $F_{rc}(X,Y) \leq F_p(X,Y)$, take arbitrary $N(K,G) \in F_{rc}(X,Y)$ with $K \subseteq X$ a C-compact subset of X and $G \subseteq Y$ an open set. Then K is finite. Taking $F = K$, $N(F,G) \in F_p(X,Y)$ and

$$N(F,G) \subseteq N(K,G).$$

Hence $N(K,G) \in F_p(X,Y)$ and so $F_{rc}(X,Y) \leq F_p(X,Y)$.

In particular, if X is discrete, then every C-compact subset of X is finite and hence $F_p(X,Y) = F_{rc}(X,Y)$. □

Remark 2.4. We can also define the t_λ-open topologies on $C(X,Y)$ by taking the collection $\{N_c(A,G) : A \in \lambda, \text{ open } G \subseteq Y\}$ as its subbase, where

$$N_c(A,G) = \{f \in C(X,Y) : f(A) \subseteq G\}.$$

In this case, Lemma 2.2 need not hold.

Example 2.5. Let $X = \mathbb{R}$ (with the usual topology), $Y = \{0,1\}$ (with the discrete topology) and λ be a family of connected subsets of X. Then $N_c([0,1],\{0\}) \subseteq N_c(\mathbb{R},\{0\})$, but $\mathbb{R} \nsubseteq [0,1]$.

However, under some additional hypotheses, an analogue of Lemma 2.2 for $C(X,Y)$ may be obtained as follows:

Proposition 2.6. *Let X be a Tychonoff (completely regular Hausdorff) space, and Y a Hausdorff topological space containing a non-trivial path $p : [0,1] \mapsto Y$ such that $p(0) \neq p(1)$, and let G and H be open sets in Y such that $p(0) \in H$ and $p(1) \notin G$. Suppose that $A, B \subseteq X$ are non-empty sets such that $N_c(B,H) \subseteq N_c(A,G)$. Then $A \subseteq \overline{B}$.*

Proof. (cf. [23], p. 5) Let $p : [0,1] \to Y$ be a path (continuous function) in Y such that $p(0) \neq p(1)$. Suppose $N_c(B,H) \subseteq N_c(A,G)$, but $A \nsubseteq \overline{B}$. Then there exists some $x_0 \in A \backslash \overline{B}$. Since X is a Tychonoff space, there exists a $\varphi \in C(X,[0,1])$ such that

$$\varphi(\overline{B}) = \{0\} \text{ and } \varphi(x_0) = 1.$$

Then $p \circ \varphi \in C(X,Y)$ with $p \circ \varphi \in N_c(B,H)$. Indeed, for any $x \in B$, $(p \circ \varphi)(x) = p(\varphi(x)) = p(0) \in H$, hence $(p \circ \varphi)(B) \subseteq H$. But $p \circ \varphi \notin N_c(A,G)$, since $x_0 \in A$ and $(p \circ \varphi)(x_0) = p(\varphi(x_0)) = p(1) \notin G$, which is a contradiction. $\square$

Remark 2.7. Let X and Y be topological spaces. If $\lambda = \{X\}$, we may consider the notion of $\{X\}$-open topology on $F(X,Y)$, denoted by t_X. However, this topology would not be of much use in most settings. In fact, if X is a compact space, then $t_X \neq t_k$, in general.

Example 2.8. Let $X = [0,1]$ and $Y = \mathbb{R}$. Then $t_X \neq t_k$ on $F([0,1],\mathbb{R})$.
The set $N(\{0\},(0,1)) \bigcap N(\{1\},(2,3)) \in t_k$, but is not in t_X.

Remark 2.9. If $\lambda = \sigma(X)$ is a family of all σ-compact subsets of X or $\lambda = \sigma_0(X)$ is a family of all countable subsets of X, we may consider the notions of $\sigma(X)$-open and $\sigma_0(X)$-open topologies on $F(X,Y)$, denoted by t_σ and t_{σ_0}, respectively. However, these topologies may not have "good" topological-algebraic properties. A space $F_\lambda(X,Y)$ may not be a topological vector space or a topological group, as is shown by the following example.

Example 2.10. Let $X = Y = \mathbb{R}$. We consider $F(X,Y)$ and $\lambda = \{\mathbb{N}$ and all finite subsets of $X\}$. Then the set $W = N(\mathbb{N},(-\pi/2,\pi/2))$ is a t_{σ_0}-open set in $F(X,Y)$. Consider the function $f(x) = arctan(x)$, $x \in \mathbb{R}$. Clearly, that $f \in W$. But we do not find a basis neighborhood B of $\mathbf{0} \in F(X,Y)$ such that $f + B \subseteq W$. It follows that the space $F_\lambda(X,Y)$ is not a topological vector space.

Remark 2.11. We mention that the situation is more useful and interesting if these topologies are considered on $C(X,Y)$ with $Y = (Y,\rho)$ a metrizable topological vector space and in particulat on $C(X) = C(X,\mathbb{R})$. Let X be a Tychonoff space and $\lambda \subseteq \mathcal{P}(X)$. In addition to the t_λ-topology on $C(X,Y)$, we can define (following the terminology of [22, 17, 29]) the notion of t_{λ^*}-topology on $C(X,Y)$ which has a subbase as the collection $\{N_c^*(A,G) : A \in \lambda$, open $G \subseteq Y\}$, where

$$N^*(A,G) = \{f \in C(X,Y) : \overline{f(A)} \subseteq G\}.$$

Here the modification $\overline{f(A)} \subseteq G$ in place of $f(A) \subseteq G$ is due to McCoy and Ntantu ([22]) who used it in order to generalize the compact-open topology to real-valued noncontinuous functions, to balance the disadvantage A is compact but $f(A)$ is not compact.

In this regard, we can also consider the topology of uniform convergence on elements of λ (the λ-topology) on $C(X,Y)$, denoted by $C_{\lambda,u}(X,Y)$, which has a base at each $f \in C(X,Y)$ as the collection $\{< f, A, \varepsilon >: A \in \lambda, \varepsilon > 0\}$, where

$$< f, A, \varepsilon >= \{g \in C(X,Y) : \sup_{x \in A} \rho(f(x), g(x)) < \varepsilon\}.$$

Theorem 2.12 ([29]). *Let X be a Tychonoff space and Y a metrizable topological vector space, and let $\lambda \subseteq \mathcal{P}(X)$.*

(a) *If $C_\lambda(X,Y) = C_{\lambda,u}(X,Y)$, then the family λ consists of C-compact sets. Conversely, if λ consists of C-compact sets, then $C_\lambda(X,Y) \leq C_{\lambda,u}(X,Y)$.*

(b) *If $C_{\lambda^*}(X,Y) = C_{\lambda,u}(X,Y)$, then the family λ consists of bounded sets. Conversely, if λ is a family consisting of bounded sets such that $\overline{A \cap W} \in \lambda$ for every functionally open (co-zero) set W with $A \cap W \neq \varnothing$, then $C_{\lambda^*}(X,Y) \leq C_{\lambda,u}(X,Y)$.*

We next give a brief account of the σ-compact-open and some related topologies on $C(X) = C(X, \mathbb{R})$ ([17]). Let $\lambda \subseteq \mathcal{P}(X)$ be any family satisfying the condition: if $A, B \in \lambda$, then there exists a $C \in \lambda$ such that $A \cup B \subseteq C$. The $\sigma(X)$-open topology (the usual σ-compact-open topology) on $C(X)$ has a subbase as the family $\{N^*_\sigma(A,G) : A \in \sigma(X), G \in \mathcal{B}\}$,where $\mathcal{B}$ is the set of bounded open intervals in $\mathbb{R}$ and we denote this space by $C_\sigma(X)$. Since, for each $f \in C(X)$, $f(\overline{A}) \subseteq \overline{f(A)}$; so that $\overline{f(\overline{A})} = \overline{f(A)}$. Hence the same topology is obtained by using $N^*_c(\overline{A}, G)$, where $A \in \sigma(X)$ and $B \in \mathcal{B}$. When $\lambda = \sigma_0(X)$, we get the countable-open topology, denoted by $C_{\sigma_0}(X)$.

For reader's convenience, we summarize some important known properties of these topologies (without proof), as follows:

Theorem 2.13 ([17]). *Let X be a Tychonoff space. Then:*

(a) $C_k(X) \leq C_\sigma(X) \leq C_{\sigma,u}(X) \leq C_u(X)$.
(b) $C_k(X) = C_\sigma(X)$ *iff every σ-compact subset of X has compact closure.*
(c) $C_\sigma(X) = C_{\sigma,u}(X)$ *iff X is pseudocompact.*
(d) $C_{\sigma,u}(X) = C_u(X)$ *iff X contains a dense σ-compact subset.*
(e) *If X is separable, then $C_{\sigma,u}(X) = C_u(X)$.*
(f) *If every countable subset of X has compact closure, then $C_{\sigma_0,u}(X) \leq C_u(X)$.*

3. Closedness and completeness in $F(X,Y)$

The results of this section are motivated by those given in [13, 25, 20] regarding the closedness and completeness of certain function subspaces in $(F(X,Y), \mathcal{U}_X)$. It is well-known (e.g., [13], p. 227-229) that, for Y a uniform space, $C(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$ but not necessarily $\mathcal{U}_p$-closed. Later, some authors also obtained variants of these results for some other classes of functions, not necessarily continuous [11, 16, 34]. In this section, we establish some results for the class $C^{\alpha}(X,Y)$ of all "α-continuous" functions from X into Y [21, 16].

We first recall necessary background for quasi-uniform spaces.

Let Y be a non-empty set. A filter $\mathcal{U}$ on $Y\times Y$ is called a **quasi-uniformity** on Y [6] if it satisfies the following conditions:

(QU_1) $\triangle(Y)=\{(y,y):y\in Y\}\subseteq U$ for all $U\in\mathcal{U}$.
(QU_2) If $U\in\mathcal{U}$, there is some $V\in\mathcal{U}$ such that $V^2\subseteq U$. (Here $V^2=V\circ V=\{(x,y)\in Y\times Y:\exists\ z\in Y \text{ such that } (x,z)\in V \text{ and } (z,y)\in V\}$.)

In this case, the pair $(Y,\mathcal{U})$ is called a **quasi-uniform space**. If, in addition, $\mathcal{U}$ satisfies the symmetry condition:

(U_3) $U\in\mathcal{U}$ implies $U^{-1}:=\{(y,x):(x,y)\in U\}\in\mathcal{U}$,

then $\mathcal{U}$ is called a **uniformity** on Y and the pair $(Y,\mathcal{U})$ is called a **uniform space**. A quasi-uniform space $(Y,\mathcal{U})$ is called **locally symmetric** if, for each $y\in Y$ and each $U\in\mathcal{U}$, there is a symmetric $V\in\mathcal{U}$ such that $V^2[y]\subseteq U[y]$ [6, 25], where $U[y]=\{z\in Y:(y,z)\in U\}$.

If $(Y,\mathcal{U})$ is a quasi-uniform space, then the collection

$$T(\mathcal{U})=\{H\subseteq Y:\text{for each } y\in H, \text{ there is } U\in\mathcal{U} \text{ such that } U[y]\subseteq H\}$$

is a topology on Y, called the **topology induced by** $\mathcal{U}$.

It is well-known that every topological space is quasi-uniformizable [35, 6] and every regular topological space is a locally symmetric quasi-uniform space [6, 25]. In view of this, for any topological space Y, we may assume, without loss of generality, that $Y=(Y,\mathcal{U})$ with $\mathcal{U}$ a quasi-uniformity. Main advantage of this assumption is that one can introduce various notions of Cauchy nets and completeness. In contrast to the case of uniform spaces, the formulation of the notion of "Cauchy net" or "Cauchy filter" in quasi-uniform spaces has been fairly complicated, and has been approached by several authors (see, e.g., the survey paper by Kunzi [19]). We shall find it convenient to restrict ourselves to the notions of "right K-Cauchy net" and "right K-complete", as in [20, 36].

Recall that a net $\{y_\alpha:\alpha\in D\}$ in a quasi-uniform space $(Y,\mathcal{U})$ is said to be **$T(\mathcal{U})$-convergent** to $y\in Y$ if, for each $U\in\mathcal{U}$, there exists an $\alpha_0\in D$ such that $y_\alpha\in U[y]$ for all $\alpha\geq\alpha_0$. A net $\{y_\alpha:\alpha\in D\}$ in Y is called a **right K-Cauchy net** provided that, for each $U\in\mathcal{U}$, there exists some $\alpha_0\in D$ such that $(y_\alpha,y_\beta)\in U$ for all $\alpha,\beta\in D$ with $\alpha\geq\beta\geq\alpha_0$. $(Y,\mathcal{U})$ is called **right K-complete** if each right K-Cauchy net is $T(\mathcal{U})$-convergent in Y (cf. [20], Lemma 1, p. 289).

We now consider the notions of quasi-uniform convergence topologies on $F(X,Y)$, which are parallel to those of the set-open topologies. Let X be a topological space and $(Y,\mathcal{U})$ a quasi-uniform space, and let $\lambda\subseteq\mathcal{P}(X)$ be a collection which covers X. For any $A\in\lambda$ and $U\in\mathcal{U}$, let

$$\widehat{U}|A=\{(f,g)\in F(X,Y)\times F(X,Y):(f(x),g(x))\in U \text{ for all } x\in A\}.$$

Then the collection $\{\widehat{U}|A : A \in \lambda \text{ and } U \in \mathcal{U}\}$ forms a subbase for a quasi-uniformity, called the **quasi-uniformity of quasi-uniform convergence on the sets in** λ induced by $\mathcal{U}$. The resultant topology on $F(X,Y)$ is called the **topology of quasi-uniform convergence on the sets in** λ [20] and is denoted $\mathcal{U}_\lambda$ (or by $F_{\lambda,\mathcal{U}}(X,Y)$ by some authors).

(1) If $\lambda = \{X\}$, $\mathcal{U}_\lambda$ is called the **quasi-uniform topology of uniform convergence** on $F(X,Y)$ and denoted by $\mathcal{U}_X$.
(2) If $\lambda = K(X)$, $\mathcal{U}_\lambda$ is called the **quasi-uniform topology of compact convergence** on $F(X,Y)$ and denoted by $\mathcal{U}_k$.
(3) If $\lambda = F(X)$, $\mathcal{U}_\lambda$ is called the **quasi-uniform topology of pointwise convergence** on $F(X,Y)$ and denoted by $\mathcal{U}_p$.

Clearly, $\mathcal{U}_p \leq \mathcal{U}_k \leq \mathcal{U}_X$.

We shall require the following result which extends ([13], Theorem 8, p. 226-227) from uniform to quasi-uniform spaces.

Lemma 3.1 ([1])**.** *Let X be a topological space and $(Y,\mathcal{U})$ a quasi-uniform space. Let $\{f_\alpha : \alpha \in D\}$ be a net in $F(X,Y)$ such that:*

(i) $\{f_\alpha : \alpha \in D\}$ is a right K-Cauchy net in $(F(X,Y),\mathcal{U}_X)$,
(ii) $f_\alpha \xrightarrow{\mathcal{U}_p} f$ on X.

Then $f_\alpha \xrightarrow{\mathcal{U}_X} f$.

Recall that a subset A of (X,τ) is called: α-**open** [26] if $A \subseteq int(cl(intA)))$. A function $f : X \to Y$ is said to be α-**continuous** [21] if $f^{-1}(H)$ is α-open in X for each open set H in Y; equivalently, if, for each point x of X and for each neighborhood H of $f(x)$, there exists an α-neighborhood G of x such that $f(G) \subset H$. Let $C^\alpha(X,Y)$ denote the set of all α-continuous functions from X into Y. It is easy to see that $C(X,Y) \subseteq C^\alpha(X,Y)$.

Theorem 3.2. *Let X be a topological space and $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space. Then:*

(a) *$C^\alpha(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.*
(b) *If Y is right K-complete, then $C^\alpha(X,Y)$ is right $\mathcal{U}_X - K$-complete.*

Proof. (a) Let $f \in F(X,Y)$ with $f \in \mathcal{U}_X - cl[C^\alpha(X,Y)]$. Let $x_0 \in X$ and H any open neighborhood of $f(x_0)$ in Y. Choose $U \in \mathcal{U}$ such that $U[f(x_0)] \subseteq H$. Choose a $V \in \mathcal{U}$ with $V = V^{-1}$ and such that

$$V^2[f(x_0)] \subseteq U[f(x_0)].$$

Choose $W \in \mathcal{U}$ such that $W^2 \subseteq V$. There exists $g \in C^\alpha(X,Y)$ such that $g \in \widehat{W}[f]$. Then we have

$$(f(y), g(y)) \in W \subseteq W^2 \subseteq V \text{ for all } y \in X.$$

By α-continuity of g at x_0, there exists an α-neighborhood G containing x_0 in X such that $g(G) \subseteq V[f(x_0)]$. Then, for any $y \in G$,

$$f(y) \in V^{-1}[g(y)] = V[g(y)] \subseteq V[V[f(x_0)]] \subseteq U[f(x_0)] \subseteq H.$$

Thus $f \in C^\alpha(X,Y)$.

(b) Suppose Y is right K-complete, and let $\{f_\alpha : \alpha \in I\}$ be a right $\mathcal{U}_X - K$-Cauchy net in $C^\alpha(X,Y)$. Let $U \in \mathcal{U}$ and let $x \in X$ be fixed. There exists $\alpha_0 \in I$ such that $(f_\alpha(y), f_\beta(y)) \in U$ for all $\alpha \geq \beta \geq \alpha_0$ and $y \in X$. In particular, $\{f_\alpha(x) : \alpha \in I\}$ is a right K-Cauchy net in Y. Consequently, we have an $f \in F(X,Y)$ such that $f_\alpha \xrightarrow{\mathcal{U}_p} f$. Then, by Lemma 3.1, $f_\alpha \xrightarrow{\mathcal{U}_X} f$, and, by part (a), $f \in C^\alpha(X,Y)$. Thus $(C^\alpha(X,Y), \mathcal{U}_X)$ is right K-complete. □

Corollary 3.3 ([16], Theorem 3.2, p. 950)**.** *Let X be a topological space and $(Y,\mathcal{U})$ a uniform space. Then:*

(a) *$C^\alpha(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.*

(b) *If Y is complete then $C^\alpha(X,Y)$ is $\mathcal{U}_X$-complete.*

Finally, we mention that the quasi-uniform topologies $\mathcal{U}_\lambda$ ($\lambda \subseteq \mathcal{P}(X)$) are, in general, different from their corresponding t_λ-open topologies ($\lambda \subseteq \mathcal{P}(X)$) even in the case of $Y = \mathbb{R}$.

Example 3.4. Let $X = \mathbb{R} = Y$, and let $\lambda = \{\mathbb{N}\}$ and $G = (-1,1)$, an open set in $Y = \mathbb{R}$. Then $N_c(A,G) = \{f \in C(\mathbb{R},\mathbb{R}) : f(\mathbb{N}) \subseteq (-1,1)\}$ is a t_λ-open set in $C(\mathbb{R},\mathbb{R})$, but it is not an open set in the topology of uniform convergence $\mathcal{U}_\lambda$ on $C(\mathbb{R},\mathbb{R})$.

For more recent contribution on the coincidence of set-open and uniform convergence topologies, the interested reader is refered to the papers [4, 29, 33].

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